Introduction

Slide 1

We propose in this paper a new point of view which unifies two well known filter families for approximating ideal fractional delay filters: *Lagrange Interpolator Filters* (LIF) and *Thiran Allpass Filters*.

We achieve this unification by approximating the ideal Fourier transform of the fractional delay according to two different Padé approximations:

- series expansions and
- continued fraction expansions.

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We prove that both approximations correspond exactly to each family.

This leads to an efficient modular implementation of LIFs.

Reference to Analog Signal

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$$\begin{tikzcd} (x_k) \arrow[d, "\text{reconstruction}"'] \arrow[r, "h^d"] & (x_{k-d}) \\ x(t) = \sum_{k \in \mathbb{Z}} x_k \mathrm{sinc}(t-k) \arrow[r, "\text{delay}"] & x(t-d) \arrow[u, "\text{sampling}"'] \end{tikzcd}$$

Ideal Filter

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- Ideal Convolution Product:
  $(x_{k-d}) = (x_k) \otimes \mathrm{sinc}(k-d)$
- Ideal Fourier Transform:
  $H^d(e^{j\omega}) = e^{-jd\omega}$
- Ideal Transfer Function: analytical extention $H^d(z)$ of $H^d(e^{j\omega})$ on the $\mathbb{C}$-plane:

$$\boxed{z^{-d}}$$

Properties of $(1+x)^d$

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$x \mapsto (1+x)^d$, is analytic on the $\mathbb{C}$-plane minus the real-semiaxis $]-\infty, -1[$, thus:

- it maps the analytic extention of $e^{-jd\omega}$ for $x = e^{-j\omega} - 1$,
- it allows a power serie expansion in the open disk of radius 1,
- it allows a continued fraction expansion on its analytic domain.

Power Series Expansion of $(1+x)^d$

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- $\forall |x| < 1$,

$$(1+x)^d = \sum_{k=0}^{+\infty} \frac{d(d-1)\dots(d-k+1)}{k!} x^k$$

Convergence Problems

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- optimal delay range for LIF of order N:

$$d \in]\frac{N-1}{2}, \frac{N+1}{2}[$$

- to stay in this range, we determine the convergence domain of $z^N H_{2N+1}^{N+d}(z)$ (resp. $z^N H_{2N}^{N+d-\frac{1}{2}}(z)$ for even orders).

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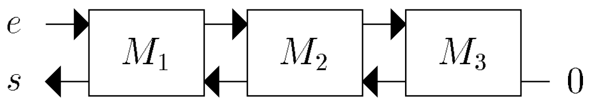

Figure 1: Quadrupole modular implementation of LIFs.

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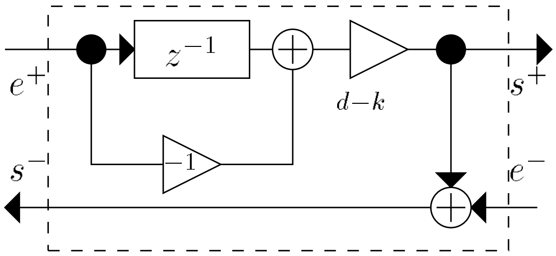

Figure 2: Detail of module k for a LIF implementation.

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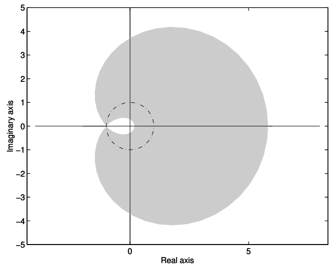
Convergence domain of LIF



The unit circle is inside the convergence domain of LIF.

FIR Filters

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$$H^d_{N+1}(z) = 1 + dw^{-1}\left(1 + \frac{d-1}{2}w^{-1}\left(\dots\right.\right.$$
$$\left.\left.\dots\left(1 + \frac{d-N}{N+1}w^{-1}\right)\dots\right)\right)$$

with $w^{-1} = z^{-1} - 1$

Link to Lagrange Interpolators

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- $H^d_N(z)$ is a polynom in z of degree N,
- maximally flat condition (see [1]):

$$\forall k = 0 \cdots N,$$
$$\frac{\partial^k}{\partial \omega^k}\left(H^d_N(e^{j\omega}) - e^{-jd\omega}\right)_{(\omega=0)} = 0$$

Continued Fraction Expansion

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- $\forall x \in \mathbb{C}\backslash]-\infty, -1[$,

$$(1+x)^d = 1 + \frac{dx}{1-}\frac{(d-1)x}{2+}\frac{(d+1)x}{3-}$$
$$\frac{(d-2)x}{2+}\frac{(d+2)x}{5-}\frac{(d-3)x}{2+}\cdots$$

(see [2])

Convergence Problems

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The continued fraction converges normally for all values exterior to the cut along the real axis from -1 to $-\infty$.

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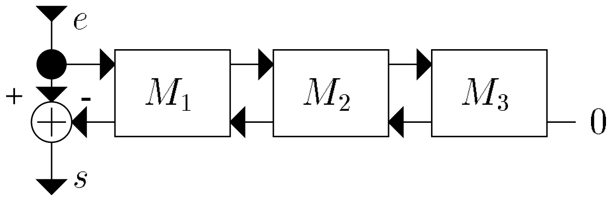

Figure 3: Quadrupole modular implementation of Allpass Fractional Delays.

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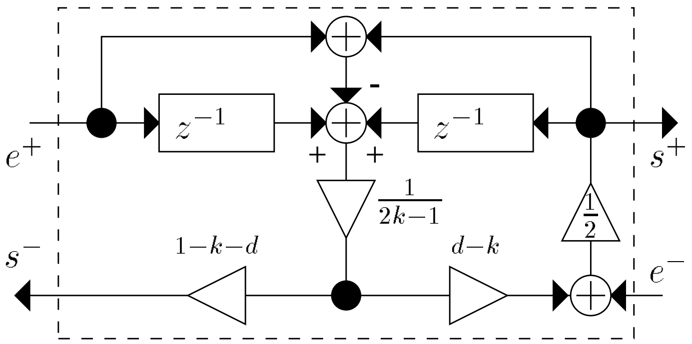

Figure 4: Detail of module k for an Allpass Fractional Delay implementation.

IIR Filters

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$$G_N^d(z) = 1 + \frac{dw^{-1}}{1-}\frac{(d-1)w^{-1}}{2+}\cdots$$
$$\cdots\frac{(d+N-1)w^{-1}}{(2N-1)-}\frac{(d-N)w^{-1}}{2},$$
with $w^{-1} = z^{-1} - 1$

Link to Thiran Allpass Filters

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- $G_N^d(z)$ is an image mirrored rational fraction in z of degree N.
- maximally flat condition (see [3]):

$$\forall k = 0 \cdots N,$$
$$\frac{\partial^k}{\partial\omega^k}\left(G_N^d(e^{j\omega}) - e^{-jd\omega}\right)_{(\omega=0)} = 0$$

Realization of a continued fraction transfer function

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If a transfer function is given by the following continued fraction:

$$H(z) = \frac{a_1 z^{-1}}{g_1 -} \frac{b_1 z^{-1}}{h_1 -} \frac{a_2 z^{-1}}{g_2 -} \frac{b_2 z^{-1}}{h_2}$$

Then the filter may be implemented using the modular structure shown on figure 1, and the module presented on figure 5.

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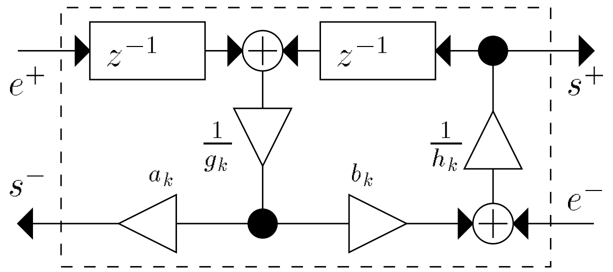

Figure 5: Detail of module k for the continued fraction implementation.

Conclusion

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We presented a new point of view for considering *Lagrange Interpolator Filters* (LIF) and *Thiran Allpass Filters* applied to Fractional Delay Digital Filters approximations.

This leads to two new closed forms for the transfer functions.

It results from the expression of the LIF a *new robust modular* and *efficient time-varying structure implementation* (see [4]).

References

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[1] E. Hermanowicz, "Explicit formulas for weighting coefficients of maximally flat tunable FIR delayers," *Electronic Letters*, vol. 28, pp. 1936–1937, Sept. 1992.

[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*. Dover publications, Inc., New York, 1972.

[3] J. P. Thiran, “Recursive digital filters with maximally flat group delay,” *IEEE Trans. Circuit Theory*, vol. CT-18, pp. 659–664, Nov. 1971.

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[4] S. Tassart and P. Depalle, “Fractional delays using Lagrange interpolators,” in *Proc. Nordical Acoustic Meeting (NAM’96)*, (Helsinki, Finland), June 1996.