USING NEURAL MODELS FOR ANALYZING TIME SERIES OF NONLINEAR DYNAMICAL SYSTEMS

Dr. Axel Röbel
GMD FIRST, Rudower Chaussee 5, 12489 Berlin, Germany

Abstract We demonstrate the use of neural networks to model and analyze time series of nonlinear dynamical systems. Based on recent results concerning the embedding of attractors from scalar time series, we use the neural models to estimate the embedding dimension and the nonnegative Lyapunov exponents of the system.

INTRODUCTION

Despite of encouraging results in nonlinear systems theory, the understanding, analysis and modeling of nonlinear processes still is one of the big challenges for future research. It is well known that the stationary behavior of nonlinear systems is characterized in state space by so called attractors. In our work we are interested in estimating the *dimension* and the *Lyapunov exponents* of these attractors directly from a time series. We do not rely on any further knowledge concerning the system dynamics.

In the following we will first give a short review on the state space reconstruction from time series and a justification for using neural networks to build time series models. Then we will be concerned with the estimation of the embedding dimension and the Lyapunov exponents and give experimental results for modeling the Lorenz equations.

RECONSTRUCTING ATTRACTORS

Assume an n-dimensional dynamical system $f(\cdot)$ evolving on an attractor A. A has fractal dimension d, which often is considerably smaller then n. The system state $\underline{z}$ is observed through a sequence of measurements $h(\underline{z})$, resulting in a time series of measurements $y_t = h(\underline{z}(t))$. Under weak assumptions concerning $h(\cdot)$ and $f(\cdot)$ the fractal embedding theorem[1] ensures that, for $D > 2d$, the set of all *delayed coordinate vectors*

$$Y_{D,\tau} = \{t > t_0 : (y_t, y_{t-\tau}, \ldots, y_{t-(D-1)\tau})\}, \tag{1}$$

with an arbitrary delay time τ, forms an embedding of A in the D-dimensional *reconstruction* space. We call the minimal D, which yields an embedding of A, the *embedding dimension* D_e. Because the embedding preserves characteristic features of A, it may be emploied for building a system model.

NEURAL MODELS

There are different topologies of neural networks that might be used for time series modeling. In our investigation we used backpropagation neural networks[3] with one hidden layer representing functions of the form

$$N(\underline{x}) = \sum_j w_j a(\sum_k w_{jk} \cdot x_k + b_j) + b_0, \tag{2}$$

where $\underline{x}$ is the delayed coordinate input vector, w and b are parameters of the network and $a(\cdot)$ is the standard sigmoid activation function. Networks of the form Eq. (2) with a finite number of hidden units are able to approximate arbitrary closely all continuous mappings $R^n \rightarrow R^1$ and its derivatives.[4] This universal approximation property is the foundation for using neural networks for time series modeling, where they are referred to as *neural models*.

Estimating embedding dimension

There exists a connection between the prediction error of the neural models, the model dimension D and the fractal dimension of the attractor d, that follows directly from the embedding theorem. For $D < D_e$ the number of ambiguous training vectors in the training set will decrease with increasing input dimension. Therefore the training error of a neural model will decrease with the input dimension reaching a minimum for $D \geq D_e$ and we may get a coarse estimate of D_e by plotting the network error with respect to D.

Estimating Lyapunov exponents

For defining the Lyapunov exponents we investigate the Jacobian of the flow of the system $D\phi_t(\underline{x}_0)$, which defines the evolution of a small disturbance Δ_0 from the starting vector $\underline{x}_0 \in A$ with evolution time t as parameter. The equivalent linearized time varying system sampled at time $k\tau$ is given as follows

$$\Delta_{k\tau} = D\phi_{k\tau}(\underline{x}_0) \cdot \Delta_0. \tag{3}$$

If $m_i(k, \underline{x}0)$ represents one of the i eigenvalues of the system in Eq. (3) the Lyapunov exponents are given by*

$$\lambda_i(\underline{x}_0) = \lim_{k\to\infty} \frac{1}{k\tau} \log_2 |m_i(k, \underline{x}_0)|. \tag{4}$$

The limit exists generally in the interesting cases[5] and is then independent of $\underline{x}_0$. The Lyapunov spectrum, $\lambda_1 > \lambda_2 > \ldots > \lambda_n$, carries valuable information concerning the system behavior.[5] A strange attractor with chaotic behavior, for example, is indicated by $\lambda_1 > 0$.

There are a number of numerical algorithms that may be used to estimate the Lyapunov spectrum from a time series.[2] The use of neural models for estimating Lyapunov exponents was first suggested by White[4].

Experimental Results

The method described above has been applied to a number of dynamical systems. In the following we present the results for the Lorenz equations with Lyapunov exponents known to be $\lambda_1 = 2.16$, $\lambda_2 = 0.0$, $\lambda_3 = -32.4$.[6]

The relation between prediction error and the model parameters D and τ is shown in Figure 1. For all choices of τ the prediction error saturates for $D \geq 3$ giving a clear indication for an embedding of the Lorenz attractor for $D \geq 3$.

The Lyapunov spectrum estimation in relation to D and $\tau = 5$ is shown in Figure 1. Obviously the estimates of the two nonnegative Lyapunov exponents are converging for $D \geq 3$, where the standard deviation covers the exact values. This supports the conclusion that the embedding dimension $D_e = 3$. The negative Lyapunov exponents, however, does not converge and sophisticated further investigation is needed to identify the $D - n$ spurious Lyapunov exponents[6].

*Note the use of $\log_2$ such that the Lyapunov exponents are measured in $Bits/s$

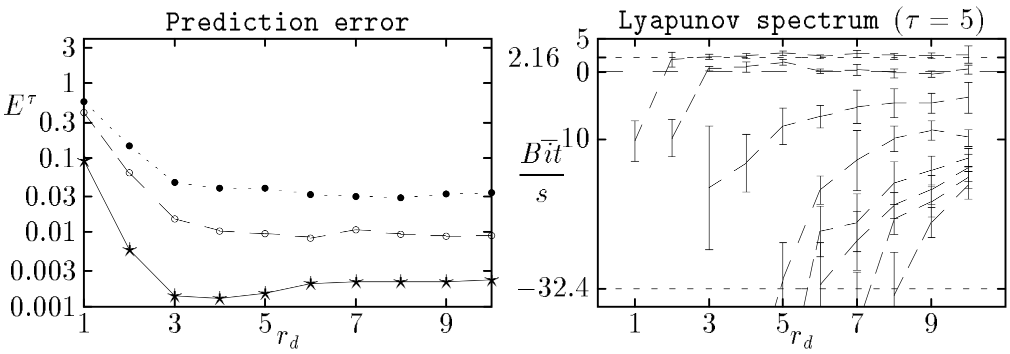

FIGURE 1. Prediction error and Lyapunov exponents of the Lorenz equation neural models for various parameters D and $\tau = \{1(\star\!\!—\!\!—),\ 5(\circ\ —),\ 9(\bullet\cdots)\}$. The known exponents of the Lorenz system are marked by horizontal lines.

CONCLUSION AND OUTLOOK

It has been shown that the neural models may be used to analyze a dynamical system directly from a scalar time series. The nonnegative Lyapunov exponents derived from the neural models are close to the exact values. And the embedding dimension may be reliable estimated. However, for estimating negative Lyapunov exponents, our systematic investigation reveals the fact, that the neural models share the same restrictions that apply to any other method employing phase space reconstruction.

REFERENCES

1. T. Sauer, J. A. Yorke and M. Casdagli. J. of Stat. Phys., 65(3/4):579--616, 1991.
2. A. S. Weigend and N. A. Gershenfeld. Time Series Prediction (Addison-Wesley, 1993).
3. A.S. Weigend, B.A. Huberman and D.E. Rumelhart. Int. J. of Neural Systems, 1, 3:193-209 (1990).
4. H. White. Neural Networks, 3, 5:535-549 (1990).
5. J.-P. Eckmann and D. Ruelle. Rev. Mod. Physics, 57, 3:617-656 (1985).
6. R. Brown, P. Bryant and H. Abarbanel. Phys. Rev. A, 43 6:2787-2806 (1991).