Neural models for estimating Lyapunov exponents and embedding dimension from time series of nonlinear dynamical systems

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1 Introduction

Despite the fact that there are a number of encouraging results in nonlinear systems theory, the understanding, analysis and modeling of nonlinear processes still is one of the big challenges for future research. Well known is the fact, known that the stationary behavior of nonlinear systems may be described in state space by means of attractors. These attractors are usually classified by their *fractal dimension*, indicating the topological complexity, and by their *Lyapunov exponents*, which measure the dynamical stability. While the estimation of the fractal dimension allows to distinguish between low dimensional and high dimensional systems or noise, the estimation of the Lyapunov exponents yields insight into the predictability of the system and may be used to decide whether the system dynamics are chaotic. In our work we are especially interested in estimating the Lyapunov exponents directly from a time series, i.e. a series of consecutive measurements of the systems output. Moreover, we will demonstrate that the embedding dimension of the attractor may be estimated by the model dimension for which the estimation of the nonnegative Lyapunov exponents does converge.
Please note, that we do not rely on any knowledge concerning the system dynamics other then a scalar time series.

In the following paper we will first give a short review on the state space reconstruction from time series and give a justification for using neural networks to build a nonlinear model from this state space reconstruction. Then we will be concerned with the estimation of the Lyapunov exponents using these models and give experimental results using two analytically known systems with chaotic behavior, which are frequently used as test systems, e.g. the Lorenz and Mackey-Glass equations.

2 Embedding Attractors

Assume an n-dimensional dynamical system $f(\cdot)$ evolving on an attractor A. The attractor has fractal dimension d, which often is considerably smaller then n. The system state $\vec{z}$ is observed through a sequence of measurements $h(\vec{z})$, resulting in a time series of measurements $y_t = h(\vec{z}(t))$. Under weak assumptions concerning $h(\cdot)$ and $f(\cdot)$ the fractal embedding theorem(Sauer *et al.*, 1991) ensures that, for $D > 2d$, the set of all *delayed coordinate vectors*

$$Y_{D,\tau} = \{t > t_0 : (y_t, y_{t-\tau}, \ldots, y_{t-(D-1)\tau})\}, \tag{1}$$

with an arbitrary delay time τ, forms an embedding of A in the D-dimensional *reconstruction* space. We call the minimal D, which yields an embedding of A, the *embedding dimension* D_e. Because an embedding preserves characteristic features of A, it may be emploied for building a system model. The problem is that we do not know the attractor dimension d or D_e and the optimal value of τ in advance.

The estimation of d and τ from a scalar time series is a crucial task(Weigend and Gershenfeld, 1993), and the results are often subject of misleading interpretation. Among other things we will demonstrate, how the Lyapunov spectrum of the neural models may be used to verify whether the dimension $D \geq D_e$ is chosen big enough, thereby leading to a coarse estimation of the fractal dimension d.

3 Neural Models

There are different topologies of neural network that might be used for time series modeling. One of the most widely know forms is the so called backpropagation neural network, which was introduced in (Rumelhart *et al.*, 1986) and first applied to a time series modeling task in (Lapedes and Farber, 1987). In the following we used backpropagation neural networks consisting of one hidden layer with sigmoid activation function and one output layer with linear activation function.
It is well known that networks of this form with a finite number of hidden units are able to approximate arbitrary closely all continuous mappings $R^n \rightarrow R^1$ and its derivatives (White, 1990). This universal approximation property suggests the use of neural networks for the time series modeling, where we we call them *neural models.*
For the following practical experiments we used a network topology with twelve hidden units and one output unit.

4 Estimating Lyapunov exponents

For defining the Lyapunov exponents we investigate the Jacobian of the flow of the system, $D\phi_t(\vec{x}_0)$, which defines the evolution of a small disturbance Δ_0 from the starting vector $\vec{x}_0$ with the evolution time t as parameter. This linear time varying system sampled at time kT is given as follows

$$\Delta_{kT} = D\phi_{kT}(\vec{x}_0) \cdot \Delta_0. \tag{2}$$

If $m_i(k, \vec{x}0)$ represents one of the i eigenvalues of the system (2) the Lyapunov exponents are given by[1]

$$\lambda_i(\vec{x}_0) = \lim_{k \to \infty} \frac{1}{kT} \log_2 |m_i(k, \vec{x}_0)|. \tag{3}$$

The limit exists generally in the interesting cases (Eckmann and Ruelle, 1985) and is then independent from $\vec{x}_0$. The sorted sequence of Lyapunov exponents

$$\lambda_1 > \lambda_2 > \ldots > \lambda_n, \tag{4}$$

called the Lyapunov spectrum, carries valuable information concerning the system behavior (Parker and Chua, 1989). An attractor with chaotic behavior, for example, is defined to have $\lambda_1 > 0$.

[1]Note the use of $\log_2$ such that the Lyapunov exponents are measured in $Bits/s$

To circumvent numerical problems the calculation of the eigenvalues from (2) is generally performed by means of a QR decomposition of $D\phi_{kT}$. This method is described in a number of references given below[2] and therefore we will not replicate the description here.

There exist a number of numerical algorithms that might be used to estimate the Lyapunov spectrum from a time series (Wolf *et al.*, 1985; Briggs, 1990). The use of neural models for estimating Lyapunov exponents was first suggested by White (1990) and investigated by Gencay and Dechert (1992) and Nychka et al. (1992) which both reported stable results. The main advantage of the latter approach is the closed analytical form of the models, such that the estimation of (2) is easy to compute. A result of the previous work is, that neural models enables us to closely estimate the Lyapunov spectrum spectrum with fairly small amount of data.

In our investigation we are interested in analyzing the impact of the reconstruction parameters D on the Lyapunov spectrum of the models. Especially we want to investigate the question, whether the negative Lyapunov exponents may be estimated from the neural models. As a result of our systematic investigation we found that, in contrast to the optimistic findings of Gencay and Dechert (1992), the negative Lyapunov exponents of the neural models are not uniquely determined.

This may be explained by the fact that the embedding ensured by the fractal embedding theorem (1), does not cover the (high-dimensional) neighborhood of the attractor. The negative Lyapunov exponents, however, are considerably depending on the system behavior in the attractor neighborhood (Eckmann and Ruelle, 1985), and therefore, reliable estimations of the Lyapunov exponents are restricted to the nonnegative part of the spectrum. Moreover, due to the higher dimension of the embedding space, compared to the original phase space, embedding techniques always produce $D-n$ spurious exponents, which in general are in the negative part of the spectrum. The problem of identifying these spurious exponents, however, requires sophisticated further investigation (Brown *et al.*, 1991).

4.1 Experimental Results

The estimation of the Lyapunov exponents from neural models has been tested using a number of different dynamical systems. In the following we present the results for the Mackey-Glass equation

$$\dot{x}(t) = \frac{a \cdot x(t - T_d)}{1 + x(t - T_d)^{10}} - b \cdot x(t). \tag{5}$$

with the parameters $a = 0.2$, $b = 0.1$ and $T_d = 30$ (Farmer, 1982), and the Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma \cdot (y - x) \\ \dot{y} &= r \cdot x - y - x \cdot z \\ \dot{z} &= x \cdot y - b \cdot z, \end{aligned} \tag{6}$$

with $\sigma = 16$, $r = 45.92$, $b = 0.1$ (Brown *et al.*, 1991). The Lorenz attractor is widely accepted to have fractal dimension $d = 2.07$ while for the Mackey-Glass attractor $d \approx 3$. The Lyapunov exponents of the Lorenz attractor are known to be $\lambda_1 = 2.16 Bit/s$, $\lambda_2 = 0.0 Bit/s$ and $\lambda_3 = -32.4 Bit/s$, while estimated values for the first three exponents of the Mackey-Glass attractor are $\lambda_1 = 0.011 Bit/s$, $\lambda_2 = 0.003 Bit/s$ and $\lambda_3 = 0.0 Bit/s$.[3] For integration of the systems we apply a second-order Runge-Kutta method with an integration time step of $dt = 0.01$ in case of the Lorenz and $dt = 0.05$ in case of the Mackey-Glass system.

[2] See for example (Eckmann and Ruelle, 1985)

[3] Note that the Mackey Glass equation describes a time delay system and therefore has infinite dimensional phase space. The Lyapunov exponents referred to are taken from (Farmer, 1982) with ±10% accuracy.

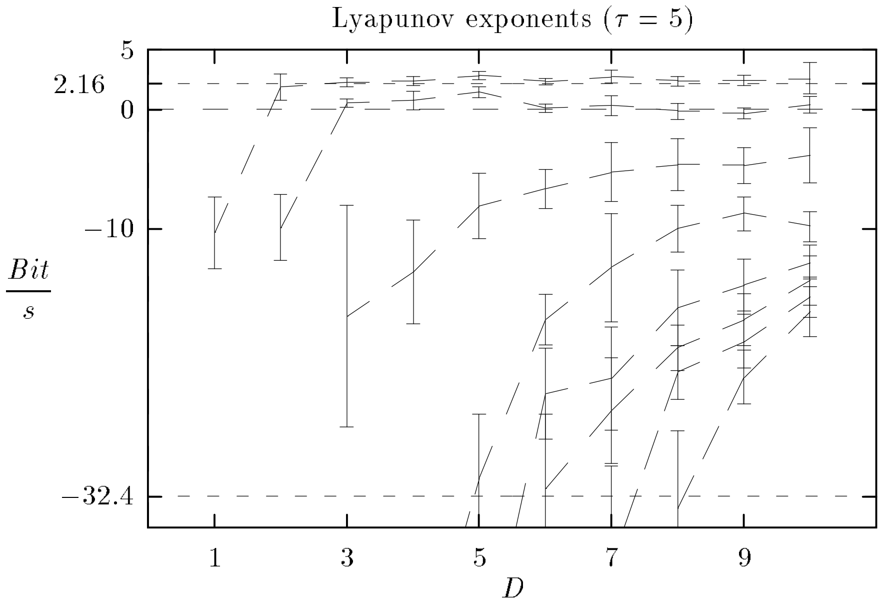

Figure 1: The average Lyapunov exponents and their standard deviation estimated from the neural models of the Lorenz equations, for various embedding dimensions D and $\tau = 5$. The exact values given in the text are indicated by horizontal lines.

For training the Lorenz system, we used the x coordinate time series with 6000 samples and $\Delta t = 0.01$. For training the Mackey-Glass system a subsampled time series with 15000 samples and $\Delta t = 1$ has been used. In both cases we optimize ten different initial weight settings for each of the time series and D and calculated the average Lyapunov exponents together with their standard deviation. The results for the Lorenz models are shown in figure (1). From the standard deviation, indicated by the vertical bars, we infer that the estimates of the two nonnegative Lyapunov exponents are converging for $D \geq 3$. From this bound we are able to estimate the embedding dimension $D_e = 3$. As expected, however, the negative exponents does not converge, which reflects the missing information concerning the neighborhood of the attractor. Similar results are obtained for the first 3 (nonnegative) Lyapunov exponents estimated from the neural models of the Mackey-Glass equation figure (2). In this case the Lyapunov exponents converged for $D \geq D_e = 7$, which is just the upper bound expected from the embedding theorem (1).
The estimation of the embedding dimension from the Lyapunov spectrum has been verified by investigating the prediction error of the various models for the Lorenz system in (Röbel, 1995) and for the Lorenz and the Mackey-Glass system in (Röbel, 1993).

5 Conclusion and outlook

It has been demonstrated that neural models may be used to analyze a dynamical system directly from a scalar time series. For the nonnegative Lyapunov exponents the neural models give numerical stable estimations. Moreover, from our experimental results, we conclude that it is possible to derive the the embedding dimension from the converging bound of the Lyapunov exponents. In contrast to previous work (Gencay and Dechert, 1992) our systematic investigation reveals the fact, that the neural models share the same

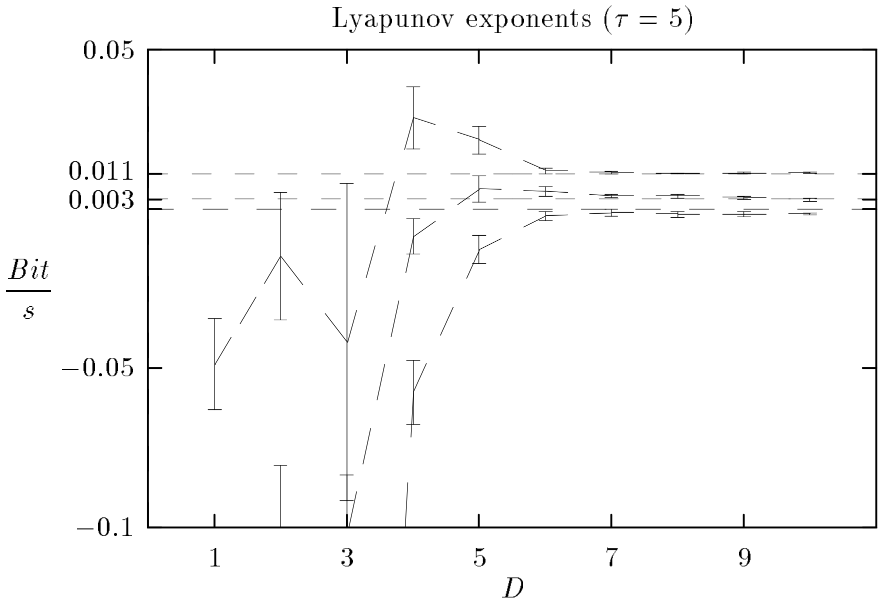

Figure 2: The average first three Lyapunov exponents and their standard deviation estimated from the neural models of the Mackey-Glass equation. The expected values given in the text are indicated by horizontal lines.

restrictions for estimating the negative Lyapunov exponents, that are common to all other known methods based on phase space reconstruction.
The application of the methods introduced in this paper have been extended to time varying systems and was applied to time series of a saxophone and a piano signal. From the Lyapunov spectrum we found that the saxophone attractor has $\lambda_1 = 0$, while the piano attractor has $\lambda_1 > 0$ and hence exhibits chaotic behavior. The music instrument models have been used to re-synthesize the training signals showing the stability of the neural models. Further results will soon be published elsewhere.

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